Computation Procedure

A quasi-linearization technique was used in the solution of the two-point boundary-value problem. The viscous solution consisted of solving ordinary differential equations [Eqs. (3) and (4)], satisfying simultaneously three conditions on the shock and four conditions on the body. Since the shock standoff distance also is unknown, a transformation to independent variable t

$$\eta = [(r_s/r_b) - 1]\, t + 1 \tag{9}$$

gives the integration range from the border to shock $0 \leq t \leq 1$, where the standoff distance $\zeta [\zeta = (r_s/r_b) - 1]$ is an unknown. The differential equation

$$d\zeta/dt = 0 \tag{10}$$

is added to Eqs. (3) and (4). Equations (3, 4, and 9) reduce to a set of seven first-order ordinary differential equations and seven boundary conditions in the viscous solution. The initial value of ζ is determined by the boundary conditions.

In conjunction with the quasi-linearization scheme, it was found necessary to reorthogonalize at every other integration step. A description of the quasi-linearization scheme is found in Ref. 5. The Gram-Schmidt orthogonalization used is described in Ref. 6.

Discussion of Results

The viscous Reynolds numbers Re used were 100, 1000, and ∞. The range of magnetic interaction parameters where solutions were possible was rather limited. At the larger magnetic interaction parameters, the quasi-linearization solution was not a solution of the original nonlinear differential equations and was not included. However, a number of conclusions can be drawn from the solutions that were obtained.

The numerical calculations of standoff distance as a function of interaction parameter are illustrated in Fig. 1. These results are in general agreement with previous works; i.e., the shock-wave standoff distance increases with increasing magnetic interaction parameter for a given viscous Reynolds number, and, for a constant magnetic interaction parameter, the standoff distance decreases with increasing viscous Reynolds number, as expected by physical reasoning.[3,4] There are possibly two features of this plot which are new. In previous works, a critical value of the interaction parameter appears at which the shock standoff distance recedes to infinity asymptotically as the applied magnetic field is increased.[3] The critical interaction parameter in the present plot no longer appears as in Ref. 3. Secondly, for $Q_m < 1$, the standoff distance increases much more rapidly for increasing Q_m in this solution than in the work by Bush.[1] If these results are applicable, it would indicate that the magnetic field, even for $Q_m < 1$, is a more effective method of controlling the hypersonic flow than previously thought.

Table 1 gives the computed values of the ratio of the resultant magnetic-field components at the shock to the dipole field at the body and also the ratio of the undistorted dipole field at the shock to that at the body. It is evident from this table that the standoff distance depends essentially upon the product of R_m and M_p, i.e., Q_m. A similar result has appeared in previous calculations.[1,3,4] One further interesting feature of the tabulated results is that the resultant radial field H_r decreases much as a dipole field, whereas the angular field component H_θ decrease is slightly greater. However, for a fixed Q_m, H_θ depends upon the relative values of R_m and M_p. It appears that the induced magnetic field is influenced mainly by the electrical conductivity. Hence a decrease in the conductivity (the induced field) should result in an H_θ more nearly approximated by a dipole field.

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An MHD Boundary-Layer Compatibility Condition

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Nomenclature

x, y	=	coordinates along and normal to surface
u, v, w	=	velocities within boundary layer
δ	=	boundary-layer thickness
η	=	y/δ, dimensionless distance
f	=	$u/U = a\eta + b\eta^2 + c\eta^3 + d\eta^4 + e\eta^5 + \ldots$
δ^*	=	$\delta \int_0^1 (1 - f) d\eta$, displacement thickness
θ	=	$\delta \int_0^1 (1 - f) d\eta$, momentum thickness
U_∞	=	undisturbed freestream velocity
τ_0	=	viscous shear stress
μ	=	fluid viscosity
P	=	pressure
ρ	=	density
σ	=	fluid electrical conductivity, assumed to be small
E	=	electric-field strength in system fixed to the fluid[1]
B_0	=	magnetic-field flux density at surface
Λ	=	$(dU/dx)\delta^2/\nu$, shape factor
Λ_m	=	$U_\infty m \delta^2/\nu$, magnetic-shape factor
mx	=	$\sigma_0 B_0^2 x/(\rho u_\infty)$, the magnetic-interaction parameter
R_N	=	$u_\infty x/\nu$, Reynolds number
ν	=	μ/ρ, kinematic viscosity

ORDINARILY, the boundary-layer equations, which represent the second law of motion and conservation of mass, are solved at every point within the boundary layer. In the von Karman-Pohlhausen method, it is assumed that it is sufficiently accurate to satisfy these equations on the average over the boundary-layer thickness. This is done by integrating the equation of motion over the boundary-layer thickness. It is this integral equation, which is satisfied then, rather than the equation of motion itself. The compatibility conditions[2] imposed in such solutions are well known. It will be shown, through the following magnetohydrodynamic (MHD) example, that these conditions are

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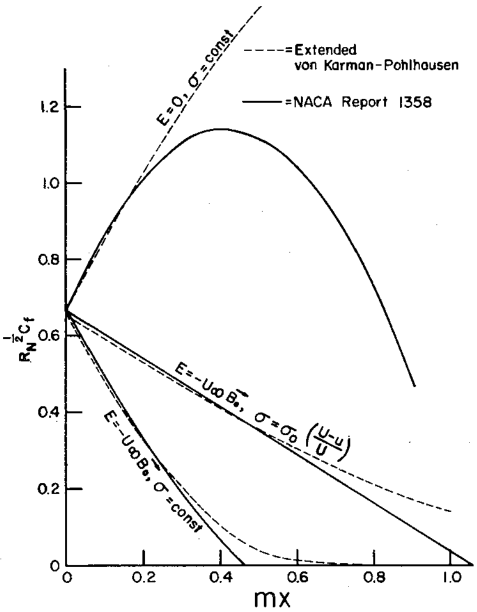

Fig 1. $R_N^{1/2} C_f$ vs mx, flat-plate flow.

insufficient in this case and that an additional compatibility condition is necessary.

The von Karman-Pohlhausen approximation will be applied to three cases of MHD flat-plate, boundary-layer flow, in which the following assumptions are made: 1) steady two-dimensional laminar flow, 2) constant fluid properties, 3) magnetic-field lines perpendicular to the surface and induced magnetic field negligible, linearizing the magnetic force term,[3] and 4) zero excess charge density. Each case will be compared to the exact solution of Rossow.[4]

Constant Conductivity $E = -U_\infty B_0$

With the definition for m, the boundary-layer equations become[3,5]

$$(\partial u/\partial x) + (\partial v/\partial y) = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + muU_\infty = -\frac{1}{\rho}\frac{dP}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad (1)$$

The equation of motion now may be integrated across the boundary-layer thickness δ to obtain[5]

$$(d/dx)(U^2\theta) + [(dU/dx) + mU_\infty]U\delta^* = \tau_0/\rho \quad (2)$$

Note that no assumptions as yet have been made as to the configuration of the two-dimensional body and its associated pressure gradient outside the boundary layer. The boundary conditions must include no slip at the wall and a continuity at the edge of the boundary layer where the inviscid flow solution is valid. Further, when adverse forces are impressed upon the flow, the possibility of an inflexion point in the velocity profile must be accounted for to allow for separation if these forces persist.

With the use of Eqs. (1) and (2) the problem may be solved in the usual way by the von Karman-Pohlhausen method. It first is assumed that the velocity profile may be approximated by a polynomial of high enough order to meet the boundary conditions and allow for the existence of an inflection point. Remaining for determination are the coefficients a, b, c, d, etc., which, in general, are functions of x. These coefficients are determined by applying boundary conditions to the equations of motion. Such a straightforward application of the von Karman-Pohlhausen approximation is inadequate for the MHD case.

In the fluid dynamic case, these boundary conditions properly introduce the pressure gradient. For the MHD case, however, there is another force affecting the velocity profile besides the pressure gradient, i.e., the ponderomotive force. Further, as can be seen from Eq. (1), this ponderomotive force is not introduced by the boundary conditions at the wall, as was the pressure gradient, because it vanishes at the wall. Thus, no MHD terms appear in the equation for the velocity profile. It is for this reason that the resulting velocity profile coefficients are identical to the fluid dynamic coefficients without MHD, even though the momentum integral does include MHD terms. If the von Karman-Pohlhausen method is to be used, a modification is necessary.

Compatibility Condition

Taking the $\partial/\partial y$ of Eq. (1) and using the continuity equation, one obtains

$$u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2} + mU_\infty\frac{\partial u}{\partial y} = \nu\frac{\partial^3 u}{\partial y^3} \quad (3)$$

The use of Eq. (3) in determining another compatibility condition or boundary condition allows the von Karman-Pohlhausen method to yield proper results. With these relations, the boundary conditions become

for $\eta = 1$

$$u = U(x) \qquad \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^3 u}{\partial y^3} = 0$$

$$-\frac{1}{\rho}\frac{dP}{dx} = U\left(\frac{dU}{dx} + mU_\infty\right)$$

for $\eta = 0$

$$u = v = 0 \qquad \nu\frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho}\frac{dP}{dx} \qquad \nu\frac{\partial^3 u}{\partial y^3} = mU_\infty\frac{\partial u}{\partial y}$$

Assuming a fifth-order polynomial for u, the relations for a, b, c, d, and e can be obtained in terms of Λ_m and Λ only. Using these relations, the displacement thickness also is found as a function of Λ_m and Λ only. The momentum integral equation then can be solved to obtain $\Lambda = \Lambda(mx)$ and $\Lambda_m = \Lambda_m(mx)$. The skin-friction coefficient $C_f = 2\tau_0/\rho U_\infty^2$

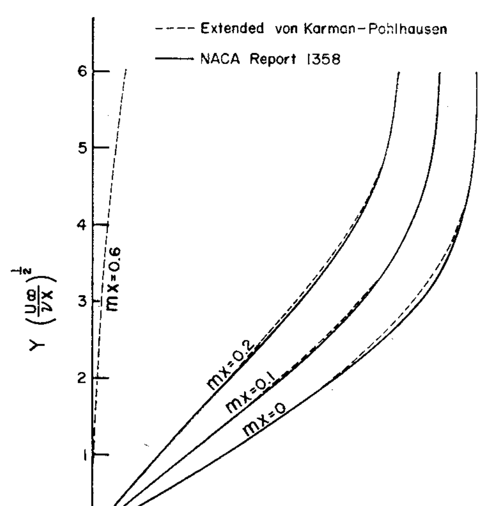

Fig. 2 Velocity profiles, $E = -U_\infty B_0$, σ = const, flat-plate flow.

also can be written as

$$R_N^{1/2}C_f = 2a(mx/\Lambda_m)^{1/2}U/U_\infty$$

and both functions u/U_∞ and $R_N^{1/2}C_f$ can be evaluated.

For the special case of a flat plate $dP/dx = 0$, $U/U_\infty = 1 - mx$, and the relations take a simpler form. Results are plotted in Figs. 1 and 2, and it can be seen that agreement with Rossow[4] is quite good for $mx < 0.2$. Further, Rossow points out that his solutions are acceptable only for $mx < 0.2$. For $mx > 0.2$ the integral solution shows the correct asymptotic behavior.

Constant Conductivity $E = 0$

The case of zero electric field may be solved in a similar manner. Since the flow outside the boundary layer is not affected by the magnetic field in this case, the pressure distribution is independent of the magnetic field. The result is the same momentum integral equation developed for the previous case. The difference lies in the fact that dU/dx is now independent of mx, which was not the case previously. For the flat plate, Λ reduces to zero.

These results are plotted in Fig. 1 and are seen to be in agreement with NACA Report 1358.[4] The limitation to small mx again is not present in the integral solution.

Variable Conductivity $E = -U_\infty B_0$

The assumption of a variable conductivity is, in many cases, more realistic. Kantrowitz[6] found that, for high Mach numbers of order 15, $\sigma = \sigma_0(U - u)/U$, where $\sigma_0 = \sigma|_{y=0}$. This assumption used by Rossow[4] therefore is used here, and the following equations are the result:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + mU_\infty(U - u)\frac{u}{U} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2} + mU_\infty\frac{\partial u}{\partial y}\left(1 - 2\frac{u}{U}\right) = \nu\frac{\partial^3 u}{\partial y^3}$$

Boundary conditions lead again to a set of equations for a, b, c, d, and e that yield the equations for δ^*/δ and θ/δ in terms of Λ and Λ_m. With these relations, the momentum integral equation may be solved for Λ and Λ_m. For a flat plate, $\Lambda = 0$ and $U = U_\infty$. Figure 1 again shows proper agreement with the exact solution.[4]

One of the principal advantages of the von Karman-Pohlhausen method is its ability to solve the boundary-layer equations once and for all in terms of parameters dependent only upon the shape of the two-dimensional body. The shape factors Λ are then known functions evaluated from the potential flow solution. Similarly, in the MHD case, the solution may be found once and for all in terms of Λ and the magnetic-shape factors Λ_m. With the introduction of the additional compatibility condition, the von Karman-Pohlhausen method often can be extended to other cases in which it previously failed. The MHD boundary-layer example selected is but one such case.

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Laminar MHD Channel Entrance Flows

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Simple parabolic velocity distribution has been used[1, 2] in applying the integral methods to solutions for magnetohydrodynamics (MHD) channel flows, but it lacks the accuracy of the solution more closely related to the one predicted by Hartmann.[3] Shohet et al.[4] and Dix[5] numerically solved the entrance-region problem by transforming the complete set of governing equations into a set of finite-difference equations. Moffatt,[3] employing the Hartmann-like velocity distribution, presented numerical solutions for pressure distributions but failed to obtain velocity fields, boundary-layer development, and friction factors. All of the forementioned authors, except those of Refs. 1 and 2, failed to obtain closed-form analytical solutions, which this author proposes to obtain with the appropriate Hartmann-like velocity distributions.

The Hartmann channel is considered here. Laminar, two-dimensional, steady, incompressible, low magnetic Reynolds-number flow is assumed, with magnetic and electric fields mutually normal to the flow direction. Hall effects are assumed to be small and thus are neglected. All of the electrical, mechanical, and thermal properties of the moving fluid or of the stationary walls will be assumed constant. Variation of the freestream velocity in the flow direction is allowed inasmuch as it satisfies the over-all conservation of mass for the constant-area channel.

With the usual boundary-layer assumptions, the momentum integral equation in the flow direction (x direction), taking into account the applied transverse magnetic-field intensity B_0, can be written as

$$\tau_\omega = \rho\,(d/dx)\,(U_\infty^2\theta) + U_\infty\,\rho\,(dU_\infty/dx) + \sigma B_0^2\delta^* \quad (1)$$

where θ and δ^* are, respectively, the momentum and the displacement thicknesses, τ_ω is the shear stress at the wall, U_∞ is the freestream velocity, and ρ and σ are the density and the electrical conductivity of the fluid.

Assuming the Hartmann-like velocity distributions, which take the form

$$U/U_\infty = \{\cosh M - \cosh M[1 - (y/\delta)]\}/(\cosh M - 1) \quad (2)$$

where $M \equiv B_0 a(\sigma/\eta)^{1/2}$ is the Hartmann number, η is the viscosity of the fluid, and δ is the boundary-layer thickness measured in the y direction, which is the direction normal to the walls, one obtains

$$\delta^*/\delta = (\sinh M/M - 1)/(\cosh M - 1) \quad (3)$$

$$\theta/\delta = [(1 + \tfrac{1}{2}\cosh M)(\sinh M/M - 2) + \tfrac{3}{2}]/(\cosh M - 1)^2 \quad (4)$$

and

$$\tau_\omega = \eta(U_\infty/\delta)[M\sinh M/(\cosh M - 1)] \quad (5)$$

Satisfying the over-all conservation of mass for the constant-area channel flow, one obtains for the dimensionless freestream velocity (i.e., ratio of freestream velocity to mean velocity)

$$U_\infty' = 1/\{1 + [(1 - \sinh M/M)/(\cosh M - 1)]\delta'\} \quad (6)$$

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